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An exact solution to the two-dimensional elasticity problem with rectangular boundaries under arbitrary edge forces

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Contents

Mathieu's approach to the fundamental problem of plane strain (but equally applicable to plane stress) with rectangular boundaries is extended so as to encompass completely arbitrary (normal and/or shear) stress distributions acting along the four edges. The method consists in breaking up the full solution into eight basic problem types which, by appropriate superposition, can be made to describe exactly the internal stress distribution arising from any imposed force distribution throughout the boundaries.

1. Introduction

The fundamental solution of two-dimensional elasticity (i.e. plane stress or plain strain) for problems with rectangular boundaries constantly recurs in many fields of applied mechanics. Solutions to those cases which are amenable to available

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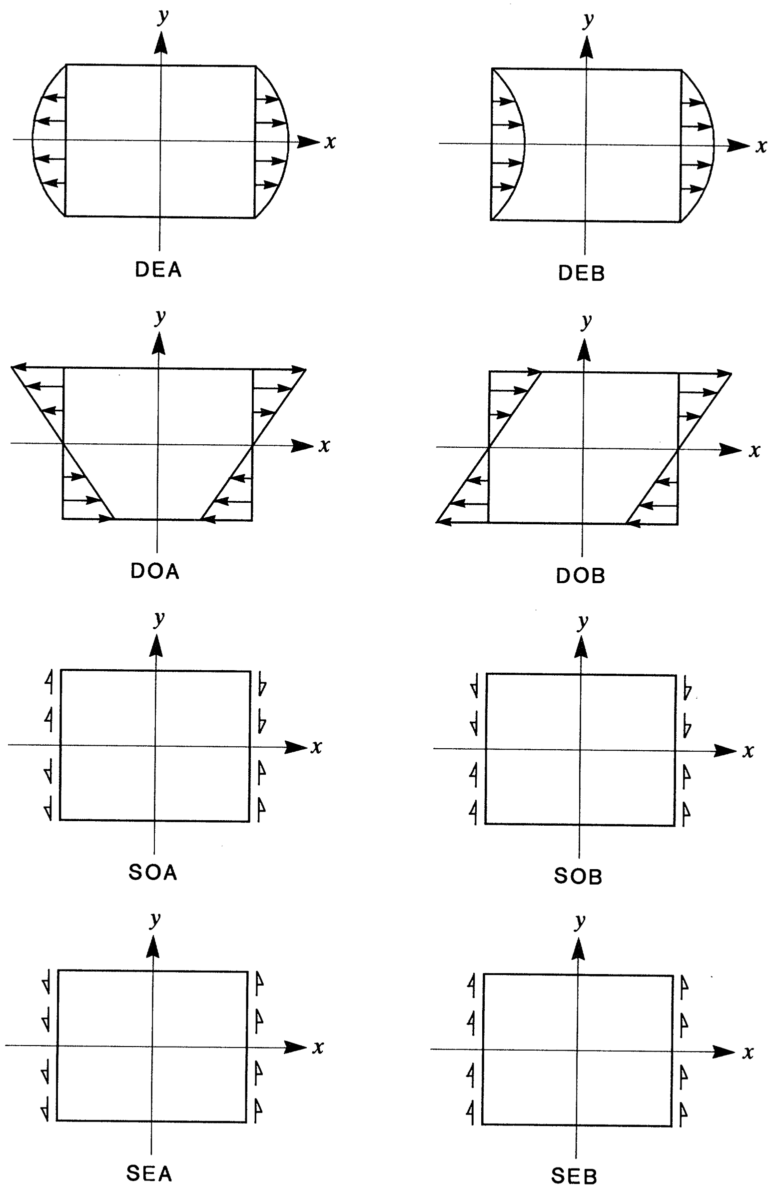

Figure 1. Stress diagrams.

analytical treatment suffer from deficiencies which arise since it is impossible to satisfy the boundary conditions (i.e. the applied direct and/or shear stresses) on all four edges; thus, for example, what is essentially Lévy's technique (Timoshenko & Goodier 1971), can only satisfy the boundary conditions on two opposite edges while the normal and shear stresses on the other two sides cannot, simultaneously, be zero (or non-zero) (Tahan 1991). On the other hand, the more adaptable numerical techniques (e.g. finite elements) suffer from the usual shortcomings associated with such methods so that, for example, difficulties may arise from steep gradients in applied and/or internal stresses. Clearly, the existence of an exact solution to two-dimensional elasticity problems with rectangular contours would be highly desirable,

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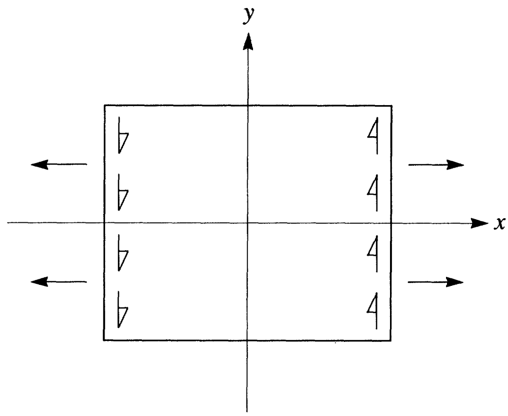

Figure 2. Sign convention for positive stresses.

both from a computational viewpoint and the automatic availability of a 'benchmark' that would serve as the standard for judging the accuracy of approximate techniques (analytical or numerical).

In view of the above, therefore, it is surprising that no attempt has been made in the past 100 years to revive Mathieu's ideas which provide the potential means for solving exactly – and without the need of simultaneous equations – any rectangular-boundary problem irrespective of the actual stress distribution on all four edges. As far as the latter is concerned, it is obvious that any applied load on one edge (direct or shear) can be divided into odd and even functions about each axis, so that the total solution can be found by the appropriate combinations of the resulting eight 'fundamental' problems. In his seminal work, Mathieu (1890) set out two basic stress problems and their solutions in terms of series for the dilatation functions. (One of these has formed the basis of several exact solutions for the buckling and vibration of non-uniformly compressed rectangular plates; for a summary of several publications in this area, see Pavlović & Baker (1988).) However, six further basic stress-problem types are required before any arbitrary problem could be solved exactly through the superposition of the eight fundamental solutions as appropriate. What is not immediately obvious is what form the series for the dilatation functions of the remaining six problems should take. To derive these, therefore, is the main object of the present paper, in which all eight fundamental solutions will be summarized in terms of dilatation functions and the ensuing displacements and stresses.

2. Problem outline

Consider a rectangular plate (or, more generally, boundary) of dimensions $a \times b$ with the origin of the axes x, y placed at its centre. Then, any stress distribution along the edges may be described in terms of the eight fundamental problems sketched in figure 1, with the usually positive sign convention (Timoshenko & Goodier 1971) given in figure 2. These refer to the edges $x = \pm\frac{1}{2}a$, but it is clear that permutation of the coordinates x and y in all the solutions will automatically cover stresses applied on the sides $y = \pm\frac{1}{2}b$. Each of the problems is defined in terms of applied stress type (D, direct stress; S, shear stress), symmetry with respect to the x-axis (E, even function; O, odd function), and symmetry with respect to the y-axis

(A, stresses are symmetrical (of equal sign); B, stresses are anti-symmetrical (of opposite sign); note that it is the relative direction, and not the tensile/compressive nature which is important). Otherwise, the applied stress distribution is, in all cases, arbitrary, its actual description being defined by the appropriate Fourier series for the loading.

Before proceeding with the solutions, it will be necessary to summarize the main governing expressions of two-dimensional elasticity (in the absence of body forces), as, in common with much of 19th-century elasticity work, Mathieu's notation and approach depart from current conventions. The two equilibrium equations (but also embodying compatibility and Hooke's law) are

$$\Delta u = -\frac{1}{\epsilon}\frac{\mathrm{d}\upsilon}{\mathrm{d}x}, \quad \Delta v = -\frac{1}{\epsilon}\frac{\mathrm{d}\upsilon}{\mathrm{d}y}, \tag{1a, b}$$

where u, v are the displacements along x and y respectively, υ is the volumetric dilatation given by

$$\upsilon = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y}, \tag{2}$$

while Δ stands for Laplace's operator. The constant ϵ is defined in terms of Lamé's parameters, i.e.

$$\epsilon = \mu/(\lambda+\mu), \tag{3}$$

these being related to the nowadays more familiar material constants, namely Young's modulus $E(=\mu(3\lambda+2\mu)(\lambda+\mu)^{-1})$ and Poisson's ratio $\nu(=\frac{1}{2}\lambda(\lambda+\mu)^{-1})$ (see, for example, Timoshenko & Goodier 1971; Sokolnikoff 1956). (In what follows, therefore, Lamé's parameters may be replaced through their equivalents, $\lambda = E\nu(1-2\nu)^{-1}(1+\nu)^{-1}$; $\mu = \frac{1}{2}E(1+\nu)^{-1}$.) Finally, by operating on expressions (1), it is easy to show that the following requirement must always be met:

$$\Delta \upsilon = 0. \tag{4}$$

Once the displacements u and v have been obtained, the stresses N_1 (direct stress along x), N_2 (direct stress along y) and T_3 (shear stress in the plane x–y) follow from the relations:

$$N_1 = \lambda\upsilon + 2\mu\frac{\mathrm{d}u}{\mathrm{d}x}, \quad N_2 = \lambda\upsilon + 2\mu\frac{\mathrm{d}v}{\mathrm{d}y}, \quad T_3 = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x}\right). \tag{5a–c}$$

Mathieu's approach is best illustrated by reference to his outline for the problem DEA. This will now be given in some detail to illustrate the procedural strategy to be followed in all eight cases. First, since series solutions will be sought, and arbitrary loadings are to be allowed for, we express the loading $f(y)$ as a Fourier series; for problem DEA,

$$f(y) = A_0 + \sum_n A_n \cos ny, \tag{6}$$

which is a half-range (even) expansion. We also split the dilatation into two components υ_1 and υ_2 such that

$$\upsilon = \upsilon_1 + \upsilon_2 \tag{7}$$

with

$$v_1 = B_0 + \sum_n B_n \cosh nx \cos ny, \quad v_2 = \beta_0 + \sum_m \beta_m \cosh my \cos mx, \tag{8a, b}$$

where

$$m = 2p\pi/a \quad \text{and} \quad n = 2q\pi/b, \tag{9a, b}$$

in which p, q are positive integers 1, 2, 3, 4, ... and the summation extends over all p, q. Expressions (8a) and (8b) satisfy separately (4), hence their sum (v) is an adequate dilatation function. It should be mentioned at this stage that other combinations for even v_1 and v_2 (e.g. $\sinh nx \sin ny$ and $\sinh my \sin mx$) would not satisfy all the necessary conditions.

Now consider a function F such that

$$\Delta F = -\epsilon^{-1} v. \tag{10}$$

Such a function satisfies (1) and (2) by putting

$$u = \frac{dF}{dx} + \alpha \int v_1 \, dx, \quad v = \frac{dF}{dy} + \alpha \int v_2 \, dx. \tag{11a, b}$$

In fact, the second term in (11) is necessary to satisfy (2), and for this condition to be fulfilled the constant α must take on the value

$$\alpha = (\lambda + 2\mu)\mu^{-1}. \tag{12}$$

Just as v was split into v_1 and v_2, F can also be divided into components F_1 and F_2 such that (10) may be rewritten as

$$\Delta F_1 = -\epsilon^{-1} v_1, \quad \Delta F_2 = -\epsilon^{-1} v_2, \tag{13a, b}$$

with the solution of these partial differential equations being expressible in terms of the series (8):

$$F_1 = -\frac{1}{2\epsilon} B_0 x^2 - \frac{1}{2\epsilon} \sum_n \frac{1}{n} B_n \, x e(nx) \cos ny + \sum_n H_n E(nx) \cos ny, \tag{14a}$$

$$F_2 = -\frac{1}{2\epsilon} \beta_0 y^2 - \frac{1}{2\epsilon} \sum_m \frac{1}{m} \beta_m \, y e(my) \cos mx + \sum_m G_m E(my) \cos mx, \tag{14b}$$

where, from now onwards,

$$E(\;) = \cosh(\;), \quad e(\;) = \sinh(\;). \tag{15a, b}$$

Using equations (5) and (11), one obtains the following expression for the stresses N_1, N_2 and T_3:

$$N_1 = \lambda v + 2\mu\alpha v_1 + 2\mu \, d^2F/dx^2, \tag{16a}$$

$$N_2 = \lambda v + 2\mu\alpha v_2 + 2\mu \, d^2F/dy^2, \tag{16b}$$

$$T_3 = \mu\left[2\frac{d^2F}{dx\,dy} + \alpha\int\frac{dv_1}{dy}\,dx + \alpha\int\frac{dv_2}{dx}\,dy\right]. \tag{16c}$$

The two additional coefficients H_n and G_m are to be determined – together with B_n and β_m – from the four boundary conditions that define the loading on the edges

$x = +\frac{1}{2}a$ and $y = +\frac{1}{2}b$. (The nature of the stress function chosen for each problem ensures the automatic satisfaction of the boundary conditions at the other two sides $x = -\frac{1}{2}a$ and $y = -\frac{1}{2}b$.) In the case DEA, these four static constraints are:

$$T_3 = 0 \quad \text{at } x = \tfrac{1}{2}a; \quad T_3 = 0 \quad \text{at } y = \tfrac{1}{2}b; \tag{17a, b}$$

$$N_1 = f(y) \quad \text{at } x = \tfrac{1}{2}a; \quad N_2 = 0 \quad \text{at } y = \tfrac{1}{2}b. \tag{18a, b}$$

Conditions (17a) and (17b) lead to the following expressions of H_n and G_m:

$$H_n = B_n\left[-\frac{1}{2n^2}+\frac{a}{4n\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right], \quad G_m = \beta_m\left[-\frac{1}{2m^2}+\frac{b}{4m\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right]. \tag{19a, b}$$

It should be noted that, to reduce expressions (17a) and (17b) so that a solution for H_n and G_m can be obtained, the choice of indices $m = 2\pi p/a$ and $n = 2\pi q/b$ was necessary just to eliminate terms in $\sin\frac{1}{2}ma$ and $\sin\frac{1}{2}nb$. By combining condition (18a), together with equation (16a) and (6), one obtains:

$$\lambda v + 2\mu\alpha v_1 + 2\mu\frac{d^2F}{dx^2} = A_0 + \sum_n A_n \cos ny, \tag{20}$$

which leads to the following expression, after some algebraic manipulations:

$$\begin{aligned}
&(\lambda+2\mu)B_0 + \lambda\beta_0 - A_0 + (\lambda+2\mu\alpha)\sum_n B_n E(\tfrac{1}{2}na)\cos ny \\
&\quad + \lambda\sum_m \beta_m E(my)\cos\tfrac{1}{2}ma - \frac{\mu}{\epsilon}\sum_n B_n\,[2E(\tfrac{1}{2}na)+\tfrac{1}{2}na\,e(\tfrac{1}{2}na)]\cos ny \\
&\quad + 2\mu\sum_n H_n n^2 E(\tfrac{1}{2}na)\cos ny + \frac{\mu}{\epsilon}\sum_m \beta_m\, my\, e(my)\cos\tfrac{1}{2}ma - 2\mu\sum_m G_m m^2 E(my)\cos\tfrac{1}{2}ma \\
&\qquad = \sum_n A_n \cos ny.
\end{aligned} \tag{21}$$

Substituting equations (19) into equation (21), we obtain the following expression:

$$\begin{aligned}
&(\lambda+\mu)^{-1}[(\lambda+2\mu)B_0+\lambda\beta_0-A_0] + \sum_n\left\{B_n\left(E(\tfrac{1}{2}na)+\tfrac{1}{2}na\frac{1}{e(\frac{1}{2}na)}\right)-\frac{A_n}{\lambda+\mu}\right\}\cos ny \\
&\quad + \sum_m \beta_m\left\{E(my)\left(1-\tfrac{1}{2}mb\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)+my\,e(my)\right\}\cos\tfrac{1}{2}ma = 0.
\end{aligned} \tag{22}$$

In the same way, condition (18b) together with equation (16b) leads to the following expression:

$$\begin{aligned}
&(\lambda+\mu)^{-1}[(\lambda+2\mu)\beta_0+\lambda B_0] + \sum_m\left\{\beta_m\left(E(\tfrac{1}{2}mb)+\tfrac{1}{2}mb\frac{1}{e(\frac{1}{2}mb)}\right)\right\}\cos mx \\
&\quad + \sum_n B_n\left\{E(nx)\left(1-\tfrac{1}{2}na\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)+nxe(nx)\right\}\cos\tfrac{1}{2}nb = 0.
\end{aligned} \tag{23}$$

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By multiplying by dy and integrating between $\pm\frac{1}{2}b$, equation (22) reduces to its first term, i.e.

$$(\lambda+\mu)^{-1}[(\lambda+2\mu)B_0+\lambda\beta_0-A_0]\,b=0. \tag{24}$$

Similarly, multiplying by dx and integrating between $\pm\frac{1}{2}a$, equation (23) reduces to its first term, i.e.

$$(\lambda+\mu)^{-1}[(\lambda+2\mu)\beta_0+\lambda B_0]\,a=0. \tag{25}$$

The solution to equations (24) and (25), therefore, leads to the following expressions for B_0 and β_0:

$$B_0=\frac{\lambda+2\mu}{4\mu(\lambda+\mu)}A_0,\quad \beta_0=-\frac{\lambda}{4\mu(\lambda+\mu)}A_0. \tag{26a, b}$$

Now, multiplying by $\cos ny\,\mathrm{d}y$ and integrating between $\pm\frac{1}{2}b$, equation (22) leads to the following expression:

$$B_n\,b\left(E(\tfrac{1}{2}na)+\tfrac{1}{2}na\frac{1}{e(\frac{1}{2}na)}\right)=\frac{b}{\lambda+\mu}A_n-8\cos\tfrac{1}{2}nb\sum_m\beta_m\frac{n^2m}{(m^2+n^2)^2}e(\tfrac{1}{2}mb)\cos\tfrac{1}{2}ma. \tag{27a}$$

Similarly, multiplying by $\cos mx\,\mathrm{d}x$ and integrating between $\pm\frac{1}{2}a$, equation (23) leads to the following expression:

$$\beta_m\,a\left(E(\tfrac{1}{2}mb)+\tfrac{1}{2}mb\frac{1}{e(\frac{1}{2}mb)}\right)=-8\cos\tfrac{1}{2}ma\sum_n B_n\frac{m^2n}{(m^2+n^2)^2}e(\tfrac{1}{2}na)\cos\tfrac{1}{2}nb. \tag{27b}$$

If we now introduce the notations

$$\tau(x)=E(x)+x/e(x)\quad\text{and}\quad\Psi(x)=e(x)/\tau(x), \tag{28a, b}$$

equations (27) yields the following expressions for B_n and β_m:

$$B_n=\frac{A_n}{(\lambda+\mu)\,\tau(\frac{1}{2}na)}-\frac{8n^2\cos\frac{1}{2}nb}{b\tau(\frac{1}{2}na)}\sum_m\beta_m\frac{m}{(m^2+n^2)^2}e(\tfrac{1}{2}mb)\cos\tfrac{1}{2}ma \tag{29a}$$

and

$$\beta_m=-\frac{8m^2\cos\frac{1}{2}ma}{a\tau(\frac{1}{2}mb)}\sum_n B_n\frac{n}{(m^2+n^2)^2}e(\tfrac{1}{2}na)\cos\tfrac{1}{2}nb. \tag{29b}$$

After recursively substituting β_m into B_n, and vice versa, rearranging summations and interchanging arbitrary summation indices, one eventually obtains the following expressions for B_n and β_m:

$$B_n=\frac{A_n}{(\lambda+\mu)\,\tau(q\pi\phi)}+\frac{16q^2(-1)^q\phi^4}{\pi^2(\lambda+\mu)\,\tau(q\pi\phi)}\sum_{q'}q'A_{q'}(-1)^{q'}\Psi(q'\pi\phi)$$
$$\times\{\Lambda_1(q,q')+(16\phi^4/\pi^2)\,\Lambda_3(q,q')+(16\phi^4/\pi^2)^2\,\Lambda_5(q,q')+(16\phi^4/\pi^2)^3\Lambda_7(q,q')+\dots\} \tag{30a}$$

and

$$\beta_m=-\frac{4p^2(-1)^p\phi}{\pi(\lambda+\mu)\,\tau(p\pi/\phi)}\sum_q qA_q(-1)^q\,\Psi(q\pi\phi)$$
$$\times\{\Lambda_0(p,q)+(16\phi^4/\pi^2)\,\Lambda_2(p,q)+(16\phi^4/\pi^2)^2\,\Lambda_4(p,q)+(16\phi^4/\pi^2)^3\,\Lambda_6(p,q)+\dots\}, \tag{30b}$$

where

$$\phi = a/b, \tag{31}$$

$$A_0(p,q) = \frac{1}{(p^2+\phi^2q^2)^2}, \quad A_1(q,q') = \sum_p \frac{p^3\Psi(p\pi/\phi)}{(p^2+\phi^2q'^2)^2} A_0(p,q), \tag{32a, b}$$

$$A_2(p,q) = \sum_{q'} \frac{q'^3\Psi(q'\pi\phi)}{(p^2+\phi^2q'^2)^2} A_1(q,q'), \quad A_3(q,q') = \sum_p \frac{p^3\Psi(p\pi/\phi)}{(p^2+\phi^2q'^2)^2} A_2(p,q), \tag{32c, d}$$

$$A_4(p,q) = \sum_{q'} \frac{q'^3\Psi(q'\pi\phi)}{(p^2+\phi^2q'^2)^2} A_3(q,q'), \quad A_5(q,q') = \sum_p \frac{p^3\Psi(p\pi/\phi)}{(p^2+\phi^2q'^2)^2} A_4(p,q), \tag{32e, f}$$

$$A_6(p,q) = \sum_{q'} \frac{q'^3\Psi(q'\pi\phi)}{(p^2+\phi^2q'^2)^2} A_5(q,q'), \quad \text{etc.}, \tag{32g}$$

and q' is clearly a 'dummy' integer variable.

Through the use of equations (11) and (14), we obtain the following equations for the displacements u and v:

$$u = B_0 x + \sum_n B_n \left\{ \left(\frac{\alpha}{2n} + \frac{a}{4\epsilon} \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) e(nx) - \frac{xE(nx)}{2\epsilon} \right\} \cos ny$$
$$+ \sum_m \beta_m \left\{ \left(\frac{1}{2m} - \frac{b}{4\epsilon} \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) + \frac{ye(my)}{2\epsilon} \right\} \sin mx \tag{33a}$$

and

$$v = \beta_0 y + \sum_n B_n \left\{ \left(\frac{1}{2n} - \frac{a}{4\epsilon} \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) + \frac{xe(nx)}{2\epsilon} \right\} \sin ny$$
$$+ \sum_m \beta_m \left\{ \left(\frac{\alpha}{2m} + \frac{b}{4\epsilon} \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) e(my) - \frac{yE(my)}{2\epsilon} \right\} \cos mx. \tag{33b}$$

Finally, equations (16) lead to the following expressions for the stress components:

$$N_1 = A_0 + (\lambda+\mu) \left\{ \sum_n B_n \left(\left(1 + \tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) - nxe(nx) \right) \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 - \tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) + my\, e(my) \right) \cos mx \right\}, \tag{34a}$$

$$N_2 = (\lambda+\mu) \left\{ \sum_n B_n \left(\left(1 - \tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) + nxe(nx) \right) \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 + \tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) - mye(my) \right) \cos mx \right\} \tag{34b}$$

and

$$T_3 = (\lambda+\mu) \left\{ \sum_n B_n \left(-\tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} e(nx) + nxE(nx) \right) \sin ny \right.$$
$$\left. + \sum_m \beta_m \left(-\tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} e(my) + myE(my) \right) \sin mx \right\}. \tag{34c}$$

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3. Choice of series and satisfaction of boundary conditions

In this section, we show that there is a unique choice of the series for the dilatations (v_1, v_2) – which will satisfy the appropriate boundary conditions and symmetry requirements of displacement and stress – for each of the eight problems. It is clear that the choice of the functional form for dilatations predetermines the subsequent form of displacement and stress. As pointed out earlier, since we only explicitly satisfy four of the eight boundary conditions (in each problem), we rely on the shape of the stress function to satisfy the remaining four automatically.

Our approach is to list the possible combinations of odd and even series in x and y, for u and v, and then to deduce which *must* be eliminated. We then determine the required form of dilatation and hence which series corresponds to each of the eight stress problems. Finally, we check that all boundary conditions are satisfied. In so doing, we note the unique relationship between the series, the mathematical sign convention for stress and the relevant physical symmetries of displacement and strain.

It is a simple matter now to determine the form of series for the dilatations for the various cases. These are shown in table 1, together with the code for the corresponding stress problems. The displacement symmetries and this correspondence with the applied stress is evident from a scrutiny of table 1. We defer detailed consideration (i.e. checking) of the boundary conditions until the solution has been presented, when the zeros of u, v and their derivatives can be identified as the restraints required for rigid-body equilibrium.

4. Complete solutions to the eight fundamental stress problems

In this section we summarize, with little comment, the main steps in the solutions for the remaining stress problems, omitting the very long and tedious algebra. We do point out the exact nature of the functions used for dilatations, based on the form specified in table 1, and in particular note why the series indices should be even (cf. $2\pi q/b$) or odd (cf. $\pi q/b$, where q is odd) as appropriate.

Henceforth, we use the following notation, as a means of compacting the expressions, in addition to that introduced earlier (i.e. expressions (28*a*) and (28*b*)):

$$\sigma(x) = \frac{e(x) - x}{E(x)}, \quad \chi(x) = \frac{E(x)}{\sigma(x)}, \tag{28c, d}$$

$$T(x) = \frac{e(x)}{E(x)}, \quad C(x) = \frac{E(x)}{e(x)}. \tag{28e, f}$$

In what follows, the eight basic problems (DEA, DEB, DOA, DOB, SOA, SOB, SEA, SEB) appear in §§4*a*–*h* respectively.

(*a*) *Problem* DEA

As this problem has been used for purposes of illustrating Mathieu's procedure, all the relevant expressions have been listed in §2. These include (17) and (18) (boundary conditions), (6) (externally applied loading), (8) and (9) (dilatation components), (14) (functions F_1 and F_2), (19) (H_n and G_m), (26), (29) and (30) (Bs and βs), (32) (Λs), (33) (displacements), and (34) (stresses).

Table 1. *Classification of problem type* (E, *even*; O, *odd*)

problem	DEA, SOB		DOB, SEA		DOA, SEB		DEB, SOA	
	x	y	x	y	x	y	x	y
v_1	E	E	O	O	E	O	O	E
v_2	E	E	O	O	E	O	O	E
F_1	E	E	O	O	E	O	O	E
F_2	E	E	O	O	E	O	O	E
u	O	E	E	O	O	O	E	E
v	E	O	O	E	E	E	O	O
N_1, N_2	E	E	O	O	E	O	O	E
T_3	O	O	E	E	O	E	E	O

(*b*) *Problem* DEB

Here

$$\left.\begin{aligned} N_1 &= f(y), \quad x = \tfrac{1}{2}a, \\ N_1 &= -f(y), \quad x = -\tfrac{1}{2}a, \\ N_2 &= 0, \quad y = \pm\tfrac{1}{2}b, \\ T_3 &= 0, \quad x = \pm\tfrac{1}{2}a \quad \text{and} \quad y = \pm\tfrac{1}{2}b, \end{aligned}\right\} \tag{4.2.1 a–d}$$

with

$$f(y) = A_0 + \sum_n A_n \cos ny. \tag{4.2.2}$$

(i) *Introduction of the series*

For a series odd in x but even in y we choose:

$$v_1 = Dx + \sum_n B_n\, e(nx) \cos ny, \quad n = \frac{2q\pi}{b}; \quad q = 1, 2, 3, \ldots, \tag{4.2.3a}$$

$$v_2 = \sum_m \beta_m\, E(my) \sin mx, \quad m = \frac{s\pi}{a}; \quad s = 1, 3, 5, \ldots. \tag{4.2.3b}$$

Hence

$$F_1 = -\frac{1}{6\epsilon} Dx^3 - \frac{1}{2\epsilon}\sum_m \frac{1}{n} B_n\, xE(nx) \cos ny + \sum_n H_n\, e(nx) \cos ny, \tag{4.2.4a}$$

$$F_2 = -\frac{1}{2\epsilon}\sum_m \frac{1}{m}\beta_m\, ye(my) \sin mx + \sum_m G_m\, E(my) \sin mx. \tag{4.2.4b}$$

(ii) *Boundary conditions*

From the conditions $T_3 = 0$ on $x = \frac{1}{2}a$ we find

$$H_n = B_n\left[-\frac{1}{2n^2} + \frac{a}{4n\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right], \tag{4.2.5a}$$

and from $T_3 = 0$ on $y = \frac{1}{2}b$, we obtain

$$G_m = \beta_m \left[-\frac{1}{2m^2} + \frac{b}{4m\epsilon} \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} \right], \tag{4.2.5b}$$

where the choice of indices n, m above is fixed to reduce expressions (4.2.1)–(4.2.5).

The first direct-stress condition, $N_1 = f(y)$ on $x = \frac{1}{2}a$, yields

$$(\lambda+\mu)^{-1}\left[\tfrac{1}{2}(\lambda+2\mu)\,Da - A_0\right] + \sum_n \left\{ B_n \left(e(\tfrac{1}{2}na) - \tfrac{1}{2}na \frac{1}{E(\frac{1}{2}na)} \right) - \frac{A_n}{\lambda+\mu} \right\} \cos ny$$
$$+ \sum_m \beta_m \left\{ E(my) \left(1 - \tfrac{1}{2}mb \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} \right) + mye(my) \right\} \sin \tfrac{1}{2}ma = 0 \tag{4.2.6a}$$

and from N_2 on $y = \frac{1}{2}b$ we have, finally,

$$\frac{\lambda Dx}{(\lambda+\mu)} + \sum_n B_n \left\{ e(nx) \left(1 - \tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) + nxE(nx) \right\} \cos \tfrac{1}{2}nb$$
$$+ \sum_m \left\{ \beta_m \left(E(\tfrac{1}{2}mb) + \tfrac{1}{2}mb \frac{1}{e(\frac{1}{2}mb)} \right) \right\} \sin mx = 0. \tag{4.2.6b}$$

(iii) *Coefficients* B_n, β_m, D

Integration of the first of the above two conditions yields

$$D = 2A_0/a(\lambda+\mu). \tag{4.2.7}$$

(This also comes from the simple requirement that constant terms, terms in x and terms in y, be separately zero.) Scaling the above by $\cos ny$ and $\sin mx$ respectively and integrating yields

$$B_n = \frac{A_n}{(\lambda+\mu)\,\sigma(\frac{1}{2}na)} - \frac{8n^2 \cos \frac{1}{2}nb}{b\sigma(\frac{1}{2}na)} \sum_m \beta_m \frac{m}{(m^2+n^2)^2} e(\tfrac{1}{2}mb) \sin \tfrac{1}{2}ma \tag{4.2.8a}$$

and

$$\beta_m = -\frac{4\lambda D}{(\lambda+\mu)\,a} \frac{\sin\frac{1}{2}ma}{m^2\tau(\frac{1}{2}mb)} - \frac{8m^2 \sin \frac{1}{2}ma}{a\tau(\frac{1}{2}mb)} \sum_n B_n \frac{n}{(m^2+n^2)^2} E(\tfrac{1}{2}na) \cos \tfrac{1}{2}nb. \tag{4.2.8b}$$

Recursive substitution leads, after rearranging the summations, to:

$$B_n = \frac{A_n}{(\lambda+\mu)\,\sigma(q\pi\phi)} + \frac{512q^2(-1)^q\phi^4}{\pi^2(\lambda+\mu)\,\sigma(q\pi\phi)} \sum_{q'} q' A_{q'} (-1)^{q'} \chi(q'\pi\phi) \{\Lambda_1(q,q')$$
$$+ (16\phi^4/\pi^2)\,\Lambda_3(q,q') + (16\phi^4/\pi^2)^2 \Lambda_5(q,q') + (16\phi^4/\pi^2)^3 \Lambda_7(q,q') + \dots\}$$
$$+ \frac{256\lambda q^2 (-1)^q A_0 \phi^3}{(\lambda+\mu)(\lambda+2\mu)\,\pi^3 \sigma(q\pi\phi)} \sum_{s'} \frac{\Psi(s'\pi/2\phi)}{s'} \{\Lambda_0(s',q)$$
$$+ (16\phi^4/\pi^2)\,\Lambda_2(s',q) + (16\phi^4/\pi^2)^2 \Lambda_4(s',q) + (16\phi^4/\pi^2)^3 \Lambda_6(s',q) + \dots\}, \tag{4.2.9}$$

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with

$$A_0(s,q) = \frac{1}{(s^2+4\phi^2q^2)^2}, \tag{4.2.10a}$$

$$A_1(q,q') = \sum_s \frac{s^3\Psi(s\pi/2\phi)}{(s^2+4\phi^2q'^2)^2} A_0(s,q), \tag{4.2.10b}$$

$$A_2(s,q) = \sum_{q'} \frac{q'^3\chi(q'\pi\phi)}{(s^2+4\phi^2q'^2)^2} A_1(q,q'), \tag{4.2.10c}$$

$$A_3(q,q') = \sum_s \frac{s^3\Psi(s\pi/2\phi)}{(s^2+4\phi^2q'^2)^2} A_2(s,q), \tag{4.2.10d}$$

$$A_4(s,q) = \sum_{q'} \frac{q'^3\chi(q'\pi\phi)}{(s^2+4\phi^2q'^2)^2} A_3(q,q'), \tag{4.2.10e}$$

$$A_5(q,q') = \sum_s \frac{s^3\Psi(s\pi/2\phi)}{(s^2+4\phi^2q'^2)^2} A_4(s,q), \tag{4.2.10f}$$

$$A_6(s,q) = \sum_{q'} \frac{q'^3\chi(q'\pi\phi)}{(s^2+4\phi^2q'^2)^2} A_5(q,q'), \tag{4.2.10g}$$

and

$$\begin{aligned}\beta_m = &-\frac{-8\lambda(-1)^s A_0}{(\lambda+\mu)(\lambda+2\mu)\pi^2 s^2\tau(s\pi/2\phi)} - \frac{4096 s^2\lambda(-1)^{(s-1)/2}\phi^4 A_0}{(\lambda+\mu)(\lambda+2\mu)\pi^4\tau(s\pi/2\phi)} \sum_{s'} \frac{\Psi(s'\pi/2\phi)}{s'} \\ &\times\{Z_1(s,s') + (16\phi^4/\pi^2)Z_2(s,s') + (16\phi^4/\pi^2)^2 Z_3(s,s') + \ldots\} \\ &-\frac{16s^2(-1)^{(s-1)/2}\phi}{\pi(\lambda+\mu)\tau(s\pi/2\phi)} \sum_{q'} A_{q'} q'(-1)^{q'}\chi(q'\pi\phi) \\ &\times\{A_0(s,q') + (16\phi^4/\pi^2)A_2(s,q') + (16\phi^4/\pi^2)^2 A_4(s,q') + \ldots\},\end{aligned} \tag{4.2.11}$$

where the remaining coefficients are

$$Z_1(s,s') = \sum_q \frac{q^3\chi(q\pi\phi)}{(s^2+4\phi^2q^2)^2} A_0(s',q), \tag{4.2.12a}$$

$$Z_2(s,s') = \sum_q \frac{q^3\chi(q\pi\phi)}{(s^2+4\phi^2q^2)^2} A_2(s',q), \tag{4.2.12b}$$

$$Z_3(s,s') = \sum_q \frac{q^3\chi(q\pi\phi)}{(s^2+4\phi^2q^2)^2} A_4(s',q). \tag{4.2.12c}$$

(iv) *Displacements*

$$\begin{aligned}u = \tfrac{1}{2}Dx^2 &+ \sum_n B_n\left\{\left(\frac{\alpha}{2n} + \frac{a}{4\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)E(nx) - \frac{xe(nx)}{2\epsilon}\right\}\cos ny \\ &+ \sum_m \beta_m\left\{\left(\frac{-1}{2m} + \frac{b}{4\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my) - \frac{ye(my)}{2\epsilon}\right\}\cos mx\end{aligned} \tag{4.2.13a}$$

and

$$v = \sum_n B_n \left\{ \left(\frac{1}{2n} - \frac{a}{4\epsilon} \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) + \frac{xE(nx)}{2\epsilon} \right\} \sin ny$$
$$+ \sum_m \beta_m \left\{ \left(\frac{\alpha}{2m} + \frac{b}{4\epsilon} \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} \right) e(my) - \frac{yE(my)}{2\epsilon} \right\} \sin mx. \quad (4.2.13\,b)$$

(v) *Stresses*

Finally, then,

$$N_1 = 2A_0/a + (\lambda+\mu) \left\{ \sum_n B_n \left(\left(1 + \tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) - nxE(nx) \right) \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 - \tfrac{1}{2}mb \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} \right) E(my) + mye(my) \right) \sin mx \right\}, \quad (4.2.14\,a)$$

$$N_2 = \frac{2\lambda A_0 x}{a(\lambda+2\mu)} + (\lambda+\mu) \left\{ \sum_n B_n \left(\left(1 - \tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) + nxE(nx) \right) \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 + \tfrac{1}{2}mb \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb} \right) E(my) - mye(my) \right) \sin mx \right\} \quad (4.2.14\,b)$$

and

$$T_3 = (\lambda+\mu) \left\{ \sum_n B_n \left(-\tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} E(nx) + nxe(nx) \right) \sin ny \right.$$
$$\left. + \sum_m \beta_m \left(\tfrac{1}{2}mb \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} e(my) + myE(my) \right) \cos mx \right\}. \quad (4.2.14\,c)$$

(*c*) *Problem* DOA

Here

$$\left. \begin{aligned} N_1 &= f(y), \quad x = \pm\tfrac{1}{2}a, \\ N_2 &= 0, \quad y = \pm\tfrac{1}{2}b, \\ T_3 &= 0, \quad x = \pm\tfrac{1}{2}a \quad \text{and} \quad y = \pm\tfrac{1}{2}b, \end{aligned} \right\} \quad (4.3.1\,a\text{–}c)$$

with

$$f(y) = \sum_n A_n \sin ny. \quad (4.3.2)$$

(i) *Introduction of the series*

$$v_1 = \sum_n B_n E(nx) \sin ny, \quad n = \frac{r\pi}{b}; \quad r = 1, 3, 5, \ldots, \quad (4.3.3\,a)$$

$$v_2 = \sum_m \beta_m e(my) \cos mx, \quad m = \frac{2\pi p}{a}; \quad p = 1, 2, 3, \ldots, \quad (4.3.3\,b)$$

with

$$F_1 = -\frac{1}{2\epsilon} \sum_n \frac{1}{n} B_n xe(nx) \sin ny + \sum_n H_n E(nx) \sin ny, \quad (4.3.4\,a)$$

$$F_2 = -\frac{1}{2\epsilon} \sum_m \frac{1}{m} \beta_m yE(my) \cos mx + \sum_m G_m e(my) \cos mx. \quad (4.3.4\,b)$$

(ii) *Boundary conditions*

From $T_3 = 0$ on $x = \frac{1}{2}a$

$$H_n = B_n\left[-\frac{1}{2n^2}+\frac{a}{4n\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right] \qquad (4.3.5a)$$

and from $T_3 = 0$ on $y = \frac{1}{2}b$

$$G_m = \beta_m\left[-\frac{1}{2m^2}+\frac{b}{4m\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right]. \qquad (4.3.5b)$$

In eliminating terms in F_1 from the first, and F_2 from the second conditions, we require $\sin\frac{1}{2}ma = \cos\frac{1}{2}nb = 0$ thus requiring the choice: $n = r\pi/b$ and $m = 2\pi p/a$.

From $N_1 = f(y)$ on $x = \frac{1}{2}a$

$$\sum_n\left\{B_n\left(E(\tfrac{1}{2}na)+\tfrac{1}{2}na\frac{1}{e(\frac{1}{2}na)}\right)-\frac{A_n}{(\lambda+\mu)}\right\}\sin ny + \sum_m\beta_m\left\{e(my)\left(1-\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)+myE(my)\right\}\cos\tfrac{1}{2}ma = 0, \qquad (4.3.6a)$$

and from N_2 on $y = \frac{1}{2}b$

$$\sum_n B_n\left\{E(nx)\left(1-\tfrac{1}{2}na\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)+nxe(nx)\right\}\sin\tfrac{1}{2}nb + \sum_m\left\{\beta_m\left(e(\tfrac{1}{2}mb)-\tfrac{1}{2}mb\frac{1}{E(\frac{1}{2}mb)}\right)\right\}\cos mx = 0. \qquad (4.3.6b)$$

(iii) *Coefficients* B_n, β_m

From (ii) above, upon the usual integration, we find

$$B_n = \frac{A_n}{(\lambda+\mu)\,\tau(\frac{1}{2}na)}-\frac{8n^2\sin\frac{1}{2}nb}{b\tau(\frac{1}{2}na)}\sum_m\beta_m\frac{m}{(m^2+n^2)^2}E(\tfrac{1}{2}mb)\cos\tfrac{1}{2}ma \qquad (4.3.7a)$$

and

$$\beta_m = -\frac{8m^2\cos\frac{1}{2}ma}{a\sigma(\frac{1}{2}mb)}\sum_n B_n\frac{n}{(m^2+n^2)^2}\,e(\tfrac{1}{2}na)\sin\tfrac{1}{2}nb. \qquad (4.3.7b)$$

Recursive substitution and extensive rearrangement of summations, with

$$\cos\tfrac{1}{2}ma = (-1)^p, \quad \sin\tfrac{1}{2}nb = (-1)^{(r-1)/2},$$

leads to

$$B_n = \frac{A_n}{(\lambda+\mu)\,\tau(\frac{1}{2}r\pi\phi)}+\frac{2r^2(-1)^{(r-1)/2}\phi^4}{\pi^2(\lambda+\mu)\,\tau(\frac{1}{2}r\pi\phi)}\sum_{r'}r'A_{r'}(-1)^{(r'-1)/2}\Psi(r'\pi\phi/2) \times\{\Lambda_1(r,r')+(2\phi^4/\pi^2)\,\Lambda_3(r,r')+(2\phi^4/\pi^2)^2\,\Lambda_5(r,r')+\ldots\} \qquad (4.3.8a)$$

and

$$\beta_m = -\frac{2p^2(-1)^p\phi}{\pi(\lambda+\mu)\,\sigma(p\pi/\phi)}\sum_r rA_r(-1)^{(r-1)/2}\,\Psi(\tfrac{1}{2}r\pi\phi) \times\{\Lambda_0(p,r)+(2\phi^4/\pi^2)\,\Lambda_2(p,r)+(2\phi^4/\pi^2)^2\,\Lambda_4(p,r)+\ldots\}, \qquad (4.3.8b)$$

where

$$\Lambda_0(p,r) = \frac{1}{(p^2+\frac{1}{4}\phi^2r^2)^2}, \tag{4.3.9a}$$

$$\Lambda_1(r,r') = \sum_p \frac{p^3\chi(p\pi/\phi)}{(p^2+\frac{1}{4}\phi^2r'^2)^2}\Lambda_0(p,r), \tag{4.3.9b}$$

$$\Lambda_2(p,r) = \sum_{r'} \frac{r'^3\Psi(\frac{1}{2}r'\pi\phi)}{(p^2+\frac{1}{4}\phi^2r'^2)^2}\Lambda_1(r,r'), \tag{4.3.9c}$$

$$\Lambda_3(r,r') = \sum_p \frac{p^3\chi(p\pi/\phi)}{(p^2+\frac{1}{4}\phi^2r'^2)^2}\Lambda_2(p,r), \tag{4.3.9d}$$

$$\Lambda_4(p,r) = \sum_{r'} \frac{r'^3\Psi(\frac{1}{2}r'\pi\phi)}{(p^2+\frac{1}{4}\phi^2r'^2)^2}\Lambda_3(r,r'). \tag{4.3.9e}$$

(iv) *Displacements*

$$u = \sum_n B_n\left\{\left(\frac{\alpha}{2n}+\frac{a}{4\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)e(nx)-\frac{xE(nx)}{2\epsilon}\right\}\sin ny$$
$$+\sum_m \beta_m\left\{\left(\frac{1}{2m}-\frac{b}{4\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my)+\frac{yE(my)}{2\epsilon}\right\}\sin mx, \tag{4.3.10a}$$

$$v = \sum_n B_n\left\{\left(-\frac{1}{2n}+\frac{a}{4\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx)-\frac{xe(nx)}{2\epsilon}\right\}\cos ny$$
$$+\sum_m \beta_m\left\{\left(\frac{\alpha}{2m}+\frac{b}{4\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)E(my)-\frac{ye(my)}{2\epsilon}\right\}\cos mx. \tag{4.3.10b}$$

(v) *Stresses*

$$N_1 = (\lambda+\mu)\left\{\sum_n B_n\left(\left(1+\tfrac{1}{2}na\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx)-nxe(nx)\right)\sin ny\right.$$
$$\left.+\sum_m \beta_m\left(\left(1-\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my)+myE(my)\right)\cos mx\right\}, \tag{4.3.11a}$$

$$N_2 = (\lambda+\mu)\left\{\sum_n B_n\left(\left(1-\tfrac{1}{2}na\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right)E(nx)+nxe(nx)\right)\sin ny\right.$$
$$\left.+\sum_m \beta_m\left(\left(1+\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my)-myE(my)\right)\cos mx\right\}, \tag{4.3.11b}$$

$$T_3 = (\lambda+\mu)\left\{\sum_n B_n\left(\tfrac{1}{2}na\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}e(nx)-nxE(nx)\right)\cos ny\right.$$
$$\left.+\sum_m \beta_m\left(-\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}E(my)+mye(my)\right)\sin mx\right\}. \tag{4.3.11c}$$

(d) *Problem* DOB

Here

$$\left.\begin{aligned} N_1 &= f(y), \quad x = +\tfrac{1}{2}a, \\ N_1 &= -f(y), \quad x = -\tfrac{1}{2}a, \\ N_2 &= 0, \quad y = \pm\tfrac{1}{2}b, \\ T_3 &= 0, \quad x = \pm\tfrac{1}{2}a \quad \text{and} \quad y = \pm\tfrac{1}{2}b, \end{aligned}\right\} \qquad (4.4.1\,a\text{–}d)$$

where

$$f(y) = \sum_n A_n \sin ny. \qquad (4.4.2)$$

(i) *Introduction of the series*

$$v_1 = \sum_n B_n\, e(nx) \sin ny, \quad n = \frac{r\pi}{b}; \quad r = 1, 3, 5, \ldots, \qquad (4.4.3\,a)$$

$$v_2 = \sum_m \beta_m\, e(my) \sin mx, \quad m = \frac{s\pi}{a}; \quad p = 1, 3, 5, \ldots, \qquad (4.4.3\,b)$$

with

$$F_1 = -\frac{1}{2\epsilon}\sum_n \frac{1}{n} B_n\, xE(nx) \sin ny + \sum_n H_n\, e(nx) \sin ny, \qquad (4.4.4\,a)$$

$$F_2 = -\frac{1}{2\epsilon}\sum_m \frac{1}{m} \beta_m\, yE(my) \sin mx + \sum_m G_m\, e(my) \sin mx. \qquad (4.4.4\,b)$$

(ii) *Boundary conditions*

From $T_3 = 0$ on $x = \frac{1}{2}a$,

$$H_n = B_n\left[-\frac{1}{2n^2} + \frac{a}{4n\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right] \qquad (4.4.5\,a)$$

and from $T_3 = 0$ on $y = \frac{1}{2}b$,

$$G_m = \beta_m\left[-\frac{1}{2m^2} + \frac{b}{4m\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right]. \qquad (4.4.5\,b)$$

In eliminating terms in F_1 for the first condition, and F_2 from the second condition, we require $\cos\frac{1}{2}ma = \cos\frac{1}{2}nb = 0$ thus necessitating the choice $n = r\pi/b$ and $m = s\pi/a$.

From the condition $N_1 = f(y)$ on $x = \frac{1}{2}a$ we find

$$\sum_n \left\{B_n\left(e(\tfrac{1}{2}na) - \tfrac{1}{2}na\frac{1}{E(\frac{1}{2}na)}\right) - \frac{A_n}{(\lambda+\mu)}\right\}\sin ny$$

$$+\sum_m \beta_m\left\{e(my)\left(1 - \tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right) + myE(my)\right\}\sin\tfrac{1}{2}ma = 0, \qquad (4.4.6\,a)$$

and from N_2 on $y = \frac{1}{2}b$,

$$\sum_n B_n \left\{ e(nx)\left(1-\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right) + nxE(nx)\right\} \sin\tfrac{1}{2}nb + \sum_m \left\{\beta_m \left(e(\tfrac{1}{2}mb) - \tfrac{1}{2}mb\frac{1}{E(\frac{1}{2}mb)}\right)\right\} \sin mx = 0. \quad (4.4.6b)$$

It is clear that neither $\sin\frac{1}{2}nb$ nor $\sin\frac{1}{2}ma$ can be zero, which again requires r, s to be odd.

(iii) *Coefficients* B_n, β_m

From the above we obtain

$$B_n = \frac{A_n}{(\lambda+\mu)\,\sigma(\frac{1}{2}na)} - \frac{8n^2 \sin\frac{1}{2}nb}{b\sigma(\frac{1}{2}na)} \sum_m \beta_m \frac{m}{(m^2+n^2)^2} E(\tfrac{1}{2}mb)\sin\tfrac{1}{2}ma, \quad (4.4.7a)$$

$$\beta_m = -\frac{8m^2\sin\frac{1}{2}ma}{a\sigma(\frac{1}{2}mb)} \sum_n B_n \frac{n}{(m^2+n^2)^2} E(\tfrac{1}{2}na)\sin\tfrac{1}{2}nb. \quad (4.4.7b)$$

Recursive substitution and rearrangement of summations, with

$$\sin\tfrac{1}{2}ma = (-1)^{(s-1)/2}, \quad \sin\tfrac{1}{2}nb = (-1)^{(r-1)/2}$$

gives

$$B_n = \frac{A_n}{(\lambda+\mu)\,\sigma(\frac{1}{2}r\pi\phi)} + \frac{64r^2(-1)^{(r-1)/2}\phi^4}{\pi^2(\lambda+\mu)\,\sigma(\frac{1}{2}r\pi\phi)} \sum_{r'} r' A_{r'} (-1)^{(r'-1)/2} \chi(\tfrac{1}{2}r'\pi\phi)$$
$$\{\Lambda_1(r,r') + (64\phi^4/\pi^2)\,\Lambda_3(r,r') + (64\phi^4/\pi^2)^2\,\Lambda_5(r,r') + \ldots\}, \quad (4.4.8a)$$

$$\beta_m = -\frac{8s^2(-1)^{(s-1)/2}\phi}{\pi(\lambda+\mu)\,\sigma(s\pi/2\phi)} \sum_r rA_r(-1)^{(r-1)/2}\chi(\tfrac{1}{2}r\pi\phi)$$
$$\times\{\Lambda_0(r,s) + (64\phi^4/\pi^2)\,\Lambda_2(r,s) + (64\phi^4/\pi^2)^2\,\Lambda_4(r,s) + \ldots\}, \quad (4.4.8b)$$

where

$$\Lambda_0(r,s) = \frac{1}{(s^2+\phi^2r^2)^2}, \quad (4.4.9a)$$

$$\Lambda_1(r,r') = \sum_s \frac{s^3\chi(s\pi/2\phi)}{(s^2+\phi^2r'^2)^2}\Lambda_0(r,s), \quad (4.4.9b)$$

$$\Lambda_2(r,s) = \sum_{r'} \frac{r'^3\chi(\frac{1}{2}r'\pi\phi)}{(s^2+\phi^2r'^2)^2}\Lambda_1(r,r'), \quad (4.4.9c)$$

$$\Lambda_3(r,r') = \sum_s \frac{s^3\chi(s\pi/2\phi)}{(s^2+\phi^2r'^2)^2}\Lambda_2(r,s), \quad (4.4.9d)$$

$$\Lambda_4(r,s) = \sum_{r'} \frac{r'^3\chi(\frac{1}{2}r'\pi\phi)}{(s^2+\phi^2r'^2)^2}\Lambda_3(r,r'). \quad (4.4.9e)$$

(iv) *Displacements*

$$u = \sum_n B_n \left\{ \left(\frac{\alpha}{2n} + \frac{a}{4\epsilon} \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) E(nx) - \frac{xe(nx)}{2\epsilon} \right\} \sin ny$$

$$+ \sum_m \beta_m \left\{ \left(-\frac{1}{2m} + \frac{b}{4\epsilon} \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) e(my) - \frac{yE(my)}{2\epsilon} \right\} \cos mx, \quad (4.4.10\,a)$$

$$v = \sum_n B_n \left\{ \left(-\frac{1}{2n} + \frac{a}{4\epsilon} \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) - \frac{xE(nx)}{2\epsilon} \right\} \cos ny$$

$$+ \sum_m \beta_m \left\{ \left(\frac{\alpha}{2m} + \frac{b}{4\epsilon} \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) E(my) - \frac{ye(my)}{2\epsilon} \right\} \sin mx. \quad (4.4.10\,b)$$

(v) *Stresses*

$$N_1 = (\lambda + \mu) \left\{ \sum_n B_n \left(\left(1 + \tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) - nxE(nx) \right) \sin ny \right.$$

$$\left. + \sum_m \beta_m \left(\left(1 - \tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) e(my) + myE(my) \right) \sin mx \right\}, \quad (4.4.11\,a)$$

$$N_2 = (\lambda + \mu) \left\{ \sum_n B_n \left(\left(1 - \tfrac{1}{2}na \frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)} \right) e(nx) + nxE(nx) \right) \sin ny \right.$$

$$\left. + \sum_m \beta_m \left(\left(1 + \tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) e(my) - myE(my) \right) \sin mx \right\}, \quad (4.4.11\,b)$$

$$T_3 = (\lambda + \mu) \left\{ \sum_n B_n \left(\tfrac{1}{2}na \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} e(nx) - nxE(nx) \right) \cos ny \right.$$

$$\left. + \sum_m \beta_m \left(-\tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} E(my) + mye(my) \right) \sin mx \right\}. \quad (4.4.11\,c)$$

(*e*) *Problem* SOA

Here

$$\left. \begin{aligned} N_1 &= 0, \quad x = \pm\tfrac{1}{2}a, \\ N_2 &= 0, \quad y = \pm\tfrac{1}{2}b, \\ T_3 &= f(y), \quad x = \pm\tfrac{1}{2}a, \\ T_3 &= 0, \quad x = \pm\tfrac{1}{2}b; \end{aligned} \right\} \quad (4.5.1\,a\text{–}d)$$

with

$$f(y) = \sum_n A_n \sin ny. \quad (4.5.2)$$

(i) *Introduction of the series*

$$v_1 = \sum_n B_n\, e(nx) \cos ny, \quad n = \frac{2q\pi}{b}; \quad q = 1, 2, 3, \ldots, \quad (4.5.3\,a)$$

$$v_2 = \sum_m \beta_m E(my) \sin mx, \quad m = \frac{s\pi}{a}; \quad s = 1, 2, 3, \ldots. \quad (4.5.3\,b)$$

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Note that we do *not* use a linear term, Dx, for v_1 (as in DEB) since we will seek to establish the shear boundary conditions and the coefficient D would vanish.

$$F_1 = -\frac{1}{2\epsilon}\sum_n \frac{1}{n} B_n\, xE(nx)\cos ny + \sum_n H_n\, e(nx)\cos ny, \tag{4.5.4a}$$

$$F_2 = -\frac{1}{2\epsilon}\sum_m \frac{1}{m}\beta_m\, ye(my)\sin mx + \sum_m G_m\, E(my)\sin mx. \tag{4.5.4b}$$

(ii) *Boundary conditions*

From $T_3 = \Sigma_n A_n \sin ny$ on $x = \frac{1}{2}a$,

$$H_n = -\frac{A_n}{2\mu n^2 E(\frac{1}{2}na)} + B_n\left[-\frac{1}{2n^2} + \frac{a}{4n\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right] \tag{4.5.5a}$$

and from $T_3 = 0$ on $y = \frac{1}{2}b$,

$$G_m = \beta_m\left[-\frac{1}{2m^2} + \frac{b}{4m\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right]. \tag{4.5.5b}$$

From $N_1 = 0$ at $x = \frac{1}{2}a$

$$\sum_n \left\{-\frac{A_n\, e(\frac{1}{2}na)}{(\lambda+\mu)E(\frac{1}{2}na)} + B_n\left(e(\frac{1}{2}na) - \tfrac{1}{2}na\frac{1}{E(\frac{1}{2}na)}\right)\right\}\cos ny$$
$$+\sum_m \beta_m\left\{E(my)\left(1-\tfrac{1}{2}mb\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right) + my\, e(my)\right\}\sin\tfrac{1}{2}ma = 0 \tag{4.5.6a}$$

and, from $N_2 = 0$ *at* $y = \frac{1}{2}b$

$$\sum_n \left\{\frac{A_n\, e(nx)}{(\lambda+\mu)E(\frac{1}{2}na)} + B_n\, e(nx)\left(1-\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right) + nxE(nx)\right\}\cos\tfrac{1}{2}nb$$
$$+\sum_m\left\{\beta_m\left(E(\tfrac{1}{2}mb) + \tfrac{1}{2}mb\frac{1}{e(\frac{1}{2}mb)}\right)\right\}\sin mx = 0. \tag{4.5.6b}$$

(iii) *Coefficients* B_n, β_m

$$B_n = \frac{A_n\, T(\frac{1}{2}na)}{(\lambda+\mu)\,\sigma(\frac{1}{2}na)} - \frac{8n^2\cos\frac{1}{2}nb}{b\sigma(\frac{1}{2}na)}\sum_m \beta_m \frac{m}{(m^2+n^2)^2}e(\tfrac{1}{2}mb)\sin\tfrac{1}{2}ma, \tag{4.5.7a}$$

$$\beta_m = -\frac{4\sin\frac{1}{2}ma}{a(\lambda+\mu)\,\tau(\frac{1}{2}mb)}\sum_n A_n\frac{n}{(m^2+n^2)}\cos\tfrac{1}{2}nb$$
$$-\frac{8m^2\sin\frac{1}{2}ma}{a\tau(\frac{1}{2}mb)}\sum_n B_n\frac{nE(\frac{1}{2}na)}{(m^2+n^2)^2}\cos\tfrac{1}{2}nb. \tag{4.5.7b}$$

From these we obtain:

$$B_n = \frac{A_n\, T(q\pi\phi)}{(\lambda+\mu)\,\sigma(q\pi\phi)} + \frac{256q^2(-1)^q\phi^4}{\pi^2(\lambda+\mu)\,\sigma(q\pi\phi)}\sum_{q'} q' A_{q'}(-1)^{q'}$$
$$\times\{\Lambda_1(q,q') + (64\phi^4/\pi^2)\,\Lambda_3(q,q') + (64\phi^4/\pi^2)^2\,\Lambda_5(q,q') + \ldots\} \tag{4.5.8a}$$

and

$$\beta_m = -\frac{8(-1)^{(s-1)/2}\phi}{\pi(\lambda+\mu)\,\tau(s\pi/2\phi)}\sum_{q'} q' A_{q'}(-1)^{q'}$$
$$\times\{A_0(s,q')+(64\phi^4/\pi^2)\,A_2(s,q')+(64\phi^4/\pi^2)^2\,A_4(s,q')+\ldots\} \quad (4.5.8b)$$

with
$$A_0(s,q') = \frac{1}{(s^2+4\phi^2q'^2)^2}\left[1+\frac{2s^2T(q'\pi\phi)\,\chi(q'\pi\phi)}{(s^2+4\phi^2q'^2)}\right], \quad (4.5.9a)$$

$$A_1(q,q') = \sum_s \frac{s\Psi(s\pi/2\phi)}{(s^2+4\phi^2q^2)^2}A_0(s,q'), \quad (4.5.9b)$$

$$A_2(s,q') = 8s^2\sum_q \frac{q^3\chi(q\pi\phi)}{(s^2+4\phi^2q^2)^2}A_1(q,q'), \quad (4.5.9c)$$

$$A_3(q,q') = \sum_s \frac{s\Psi(s\pi/2\phi)}{(s^2+4\phi^2q^2)^2}A_2(s,q'), \quad (4.5.9d)$$

$$A_4(s,q') = 8s^2\sum_q \frac{q^3\chi(q\pi\phi)}{(s^2+4\phi^2q^2)^2}A_3(q,q'). \quad (4.5.9e)$$

(iv) *Displacements*

$$u = \sum_n\left\{-\frac{A_nE(nx)}{2\mu nE(\frac{1}{2}na)}+B_n\left(\frac{\alpha}{2n}+\frac{a}{4\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)E(nx)-\frac{xe(nx)}{2\epsilon}\right\}\cos ny$$
$$+\sum_m\beta_m\left\{\left(-\frac{1}{2m}+\frac{b}{4\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my)-\frac{ye(my)}{2\epsilon}\right\}\cos mx \quad (4.5.10a)$$

and

$$v = \sum_n\left\{\frac{A_ne(nx)}{2\mu nE(\frac{1}{2}na)}+B_n\left(\frac{1}{2n}-\frac{a}{4\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx)+\frac{xE(nx)}{2\epsilon}\right\}\sin ny$$
$$+\sum_m\beta_m\left\{\left(\frac{\alpha}{2m}+\frac{b}{4\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)e(my)-\frac{yE(my)}{2\epsilon}\right\}\sin mx. \quad (4.5.10b)$$

(v) *Stresses*

$$N_1 = (\lambda+\mu)\left\{\sum_n\left[-\frac{A_ne(nx)}{(\lambda+\mu)E(\frac{1}{2}na)}+B_n\left(\left(1+\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx)-nxE(nx)\right)\right]\cos ny\right.$$
$$\left.+\sum_m\beta_m\left(\left(1-\tfrac{1}{2}mb\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my)+mye(my)\right)\sin mx\right\}, \quad (4.5.11a)$$

$$N_2 = (\lambda+\mu)\left\{\sum_n\left[\frac{A_ne(nx)}{(\lambda+\mu)E(\frac{1}{2}na)}+B_n\left(\left(1-\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx)+nxE(nx)\right)\right]\cos ny\right.$$
$$\left.+\sum_m\beta_m\left(\left(1+\tfrac{1}{2}mb\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right)E(my)-mye(my)\right)\sin mx\right\} \quad (4.5.11b)$$

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and

$$T_3 = (\lambda+\mu)\left\{\sum_n\left[\frac{A_n E(nx)}{(\lambda+\mu)E(\frac{1}{2}na)}+B_n\left(-\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}E(nx)+nxe(nx)\right)\right]\sin ny\right.$$
$$\left.+\sum_m \beta_m\left(\tfrac{1}{2}mb\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}e(my)-myE(my)\right)\cos mx\right\}. \quad (4.5.11\,c)$$

(*f*) *Problem* SOB

Here

$$\left.\begin{aligned} N_1 &= 0, \quad x=\pm\tfrac{1}{2}a,\\ N_2 &= 0, \quad y=\pm\tfrac{1}{2}b,\\ T_3 &= f(y), \quad x=\tfrac{1}{2}a,\\ T_3 &= -f(y), \quad x=-\tfrac{1}{2}a, \end{aligned}\right\} \quad (4.6.1\,a\text{–}d)$$

where
$$f(y)=\sum_n A_n \sin ny. \quad (4.6.2)$$

(i) *Introduction of the series*

For dilatation, split into components v_1 and v_2, we write:

$$v_1=\sum_n B_n E(nx)\cos ny, \quad n=\frac{2q\pi}{b}; \quad q=1,2,3,\ldots, \quad (4.6.3\,a)$$

$$v_2=\sum_m \beta_m E(my)\cos mx, \quad m=\frac{2p\pi}{a}; \quad p=1,2,3,\ldots. \quad (4.6.3\,b)$$

It should be noted that, even though we use an even–even series for dilatation, we omit the constants B_0 and β_0 as these would vanish under the cross-derivatives required for shear. Thus

$$F_1=-\frac{1}{2\epsilon}\sum_n\frac{1}{n}B_n xe(nx)\cos ny+\sum_n H_n E(nx)\cos ny, \quad (4.6.4\,a)$$

$$F_2=-\frac{1}{2\epsilon}\sum_m\frac{1}{m}\beta_m ye(my)\cos mx+\sum_m G_m E(my)\cos mx. \quad (4.6.4\,b)$$

(ii) *Boundary conditions*

From the condition $T_3=f(y)$ at $x=\frac{1}{2}a$, we find

$$H_n=-\frac{A_n}{2\mu n^2 e(\frac{1}{2}na)}+B_n\left[-\frac{1}{2n^2}+\frac{a}{4n\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right] \quad (4.6.5\,a)$$

noting that $n=2\pi q/b$ was required in the elimination simply so that $\sin\frac{1}{2}nb\rightarrow 0$. The second condition gives

$$G_m=\beta_m\left[-\frac{1}{2m^2}+\frac{b}{4m\epsilon}\frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)}\right]. \quad (4.6.5\,b)$$

The conditions $N_1 = 0$ and $N_2 = 0$ give respectively:

$$\sum_n \left\{ -\frac{A_n E(\frac{1}{2}na)}{(\lambda+\mu)\, e(\frac{1}{2}na)} + B_n \left(E(\tfrac{1}{2}na) + \tfrac{1}{2}na \frac{1}{e(\frac{1}{2}na)} \right) \right\} \cos ny$$
$$+ \sum_m \beta_m \left\{ E(my) \left(1 - \tfrac{1}{2}mb \frac{E(\frac{1}{2}mb)}{e(\frac{1}{2}mb)} \right) + mye(my) \right\} \cos \tfrac{1}{2}ma = 0 \quad (4.6.6a)$$

and

$$\sum_n \left\{ \frac{A_n E(nx)}{(\lambda+\mu)\, e(\frac{1}{2}na)} + B_n \left(E(nx) \left(1 - \tfrac{1}{2}na \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} \right) + nxe(nx) \right) \right\} \cos \tfrac{1}{2}nb$$
$$+ \sum_m \left\{ \beta_m \left(E(\tfrac{1}{2}mb) + \tfrac{1}{2}mb \frac{1}{e(\frac{1}{2}mb)} \right) \right\} \cos mx = 0. \quad (4.6.6b)$$

(iii) *Coefficients* B_n, β_m

$$B_n = \frac{A_n C(\frac{1}{2}na)}{(\lambda+\mu)\, \tau(\frac{1}{2}mb)} - \frac{8n^2 \cos \frac{1}{2}nb}{b\tau(\frac{1}{2}na)} \sum_m \beta_m \frac{m}{(m^2+n^2)^2} e(\tfrac{1}{2}mb) \cos \tfrac{1}{2}ma, \quad (4.6.7a)$$

$$\beta_m = -\frac{4 \cos \frac{1}{2}ma}{a(\lambda+\mu)\, \tau(\frac{1}{2}mb)} \sum_n A_n \frac{n}{(m^2+n^2)} \cos \tfrac{1}{2}nb$$
$$- \frac{8m^2 \cos \frac{1}{2}ma}{a\tau(\frac{1}{2}mb)} \sum_n B_n \frac{n}{(m^2+n^2)^2} e(\tfrac{1}{2}na) \cos \tfrac{1}{2}nb, \quad (4.6.7b)$$

from which one obtains, recursively,

$$B_n = \frac{A_n C(q\pi\phi)}{a(\lambda+\mu)\, \tau(q\pi\phi)} - \frac{8q^2(-1)^q \phi^4}{\pi^2(\lambda+\mu)\, \tau(q\pi\phi)} \sum_{q'} q' A_{q'} (-1)^{q'}$$
$$\times \{ \Lambda_1(q,q') + (16\phi^4/\pi^2)\, \Lambda_3(q,q') + (16\phi^4/\pi^2)^2\, \Lambda_5(q,q') + \ldots \} \quad (4.6.8a)$$

and

$$\beta_m = -\frac{2(-1)^p \phi}{\pi(\lambda+\mu)\, \tau(p\pi/\phi)} \sum_{q'} q' A_{q'} (-1)^{q'}$$
$$\times \{ \Lambda_0(p,q') + (16\phi^4/\pi^2)\, \Lambda_2(p,q') + (16\phi^4/\pi^2)^2\, \Lambda_4(p,q') + \ldots \}, \quad (4.6.8b)$$

where

$$\Lambda_0(p,q') = \frac{1}{(p^2+\phi^2 q'^2)} \left[1 + \frac{2p^2 \Psi(q'\pi\phi)\, C(q'\pi\phi)}{(p^2+\phi^2 q'^2)} \right], \quad (4.6.9a)$$

$$\Lambda_1(q,q') = \sum_p \frac{p\Psi(p\pi/\phi)}{(p^2+\phi^2 q^2)^2} \Lambda_0(p,q'), \quad (4.6.9b)$$

$$\Lambda_2(p,q') = p^2 \sum_q \frac{q^3 \Psi(q\pi\phi)}{(p^2+\phi^2 q^2)^2} \Lambda_1(q,q'), \quad (4.6.9c)$$

$$\Lambda_3(q,q') = \sum_p \frac{p\Psi(p\pi/\phi)}{(p^2+\phi^2 q^2)^2} \Lambda_2(p,q'), \quad (4.6.9d)$$

$$\Lambda_4(p,q') = p^2 \sum_q \frac{q^3 \Psi(q\pi\phi)}{(p^2+\phi^2 q^2)^2} \Lambda_3(q,q'). \quad (4.6.9e)$$

(iv) *Displacements*

$$u = \sum_n \left\{ -\frac{A_n\, e(nx)}{2\mu n e(\tfrac{1}{2}na)} + B_n \left(\left(\frac{\alpha}{2n} + \frac{a}{4\epsilon}\frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) e(nx) - \frac{xE(nx)}{2\epsilon} \right) \right\} \cos ny$$
$$+ \sum_m \beta_m \left\{ \left(\frac{1}{2m} - \frac{b}{4\epsilon}\frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) + \frac{ye(my)}{2\epsilon} \right\} \sin mx, \qquad (4.6.10a)$$

$$v = \sum_n \left\{ \frac{A_n\, E(nx)}{2\mu n e(\tfrac{1}{2}na)} + B_n \left(\left(\frac{1}{2n} - \frac{a}{4\epsilon}\frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) + \frac{xe(nx)}{2\epsilon} \right) \right\} \sin ny$$
$$+ \sum_m \beta_m \left\{ \left(\frac{\alpha}{2m} + \frac{b}{4\epsilon}\frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) e(my) - \frac{yE(my)}{2\epsilon} \right\} \cos mx. \qquad (4.6.10b)$$

(v) *Stresses*

$$N_1 = (\lambda+\mu) \left\{ \sum_n \left[-\frac{A_n E(nx)}{(\lambda+\mu)\, e(\tfrac{1}{2}na)} + B_n \left(\left(1 + \tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) - nxe(nx) \right) \right] \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 - \tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) + mye(my) \right) \cos mx \right\}, \qquad (4.6.11a)$$

$$N_2 = (\lambda+\mu) \left\{ \sum_n \left[\frac{A_n E(nx)}{(\lambda+\mu)\, e(\tfrac{1}{2}na)} + B_n \left(\left(1 - \tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) E(nx) + nxe(nx) \right) \right] \cos ny \right.$$
$$\left. + \sum_m \beta_m \left(\left(1 + \tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} \right) E(my) - mye(my) \right) \cos mx \right\} \qquad (4.6.11b)$$

and

$$T_3 = (\lambda+\mu) \left\{ \sum_n \left[\frac{A_n e(nx)}{(\lambda+\mu)\, e(\tfrac{1}{2}na)} + B_n \left(-\tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} e(nx) + nxE(nx) \right) \right] \sin ny \right.$$
$$\left. + \sum_m \beta_m \left(-\tfrac{1}{2}mb \frac{E(\tfrac{1}{2}mb)}{e(\tfrac{1}{2}mb)} e(my) + myE(my) \right) \sin mx \right\}. \qquad (4.6.11c)$$

(g) *Problem* SEA

Here

$$\left.\begin{aligned} N_1 &= 0, \quad x = \pm\tfrac{1}{2}a, \quad y = \pm\tfrac{1}{2}b, \\ N_2 &= 0, \quad x = \pm\tfrac{1}{2}a, \quad y = \pm\tfrac{1}{2}b, \\ T_3 &= f(y), \quad x = \pm\tfrac{1}{2}a, \\ T_3 &= 0, \quad y = \pm\tfrac{1}{2}b, \end{aligned}\right\} \qquad (4.7.1\,a\text{–}d)$$

where

$$f(y) = \sum_n A_n \cos ny. \qquad (4.7.2)$$

(i) *Introduction of the series*

$$v_1 = \sum_n B_n\, e(nx) \sin ny, \quad n = \frac{r\pi}{b}; \quad r = 1, 3, 5, \ldots, \qquad (4.7.3a)$$

$$v_2 = \sum_n \beta_m\, e(my) \sin mx, \quad m = \frac{s\pi}{a}; \quad s = 1, 3, 5, \ldots, \qquad (4.7.3b)$$

with

$$F_1 = -\frac{1}{2\epsilon}\sum_n \frac{1}{n} B_n\, xE(nx)\sin ny + \sum_n H_n\, e(nx)\sin ny, \tag{4.7.4a}$$

$$F_2 = -\frac{1}{2\epsilon}\sum_m \frac{1}{m} \beta_m\, yE(my)\sin mx + \sum_m G_m\, e(my)\sin mx. \tag{4.7.4b}$$

(ii) *Boundary conditions*

From $T_3 = f(y)$ on $x = \frac{1}{2}a$ we find

$$H_n = \frac{A_n}{2n^2\mu E(\frac{1}{2}na)} + B_n\left[-\frac{1}{2n^2} + \frac{a}{4n\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right] \tag{4.7.5a}$$

and from $T_3 = 0$ on $y = \frac{1}{2}b$ we find G_m as for DOA, DOB:

$$G_m = \beta_m\left[-\frac{1}{2m^2} + \frac{b}{4m\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right]. \tag{4.7.5b}$$

From $N_1 = 0$ on $x = \frac{1}{2}a$ we find

$$\sum_n\left\{B_n\left(e(\tfrac{1}{2}na) - \tfrac{1}{2}na\frac{1}{E(\frac{1}{2}na)}\right) + \frac{A_n\, e(\frac{1}{2}na)}{(\lambda+\mu)\, E(\frac{1}{2}na)}\right\}\sin ny$$
$$+\sum_m \beta_m\left\{e(my)\left(1 - \tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right) + myE(my)\right\}\sin\tfrac{1}{2}ma = 0 \tag{4.7.6a}$$

and finally from N_2 on $y = \frac{1}{2}b$

$$\sum_n\left\{B_n\left(e(nx)\left(1 - \tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right) + nxE(nx)\right) - \frac{A_n\, e(nx)}{(\lambda+\mu)\, E(\frac{1}{2}na)}\right\}\sin\tfrac{1}{2}nb$$
$$+\sum_m\left\{\beta_m\left(e(\tfrac{1}{2}mb) - \tfrac{1}{2}mb\frac{1}{E(\frac{1}{2}mb)}\right)\right\}\sin mx = 0. \tag{4.7.6b}$$

(iii) *Coefficients* B_n, β_m

$$B_n = -\frac{A_n\, T(\frac{1}{2}na)}{(\lambda+\mu)\,\sigma(\frac{1}{2}na)} - \frac{8n^2\sin\frac{1}{2}nb}{b\sigma(\frac{1}{2}na)}\sum_m \beta_m \frac{m}{(m^2+n^2)^2}E(\tfrac{1}{2}mb)\sin\tfrac{1}{2}ma, \tag{4.7.7a}$$

$$\beta_m = \frac{4\sin\frac{1}{2}ma}{a\sigma(\frac{1}{2}mb)}\sum_n n\sin\tfrac{1}{2}nb\left\{\frac{A_n}{(\lambda+\mu)\,(m^2+n^2)} - \frac{2m^2B_n}{(m^2+n^2)^2}E(\tfrac{1}{2}na)\right\}. \tag{4.7.7b}$$

Recursive substitution leads to

$$B_n = -\frac{A_n\, T(\frac{1}{2}r\pi\phi)}{(\lambda+\mu)\,\sigma(\frac{1}{2}r\pi\phi)} - \frac{32r^2(-1)^{(r-1)/2}\phi^4}{\pi^2(\lambda+\mu)\,\sigma(\frac{1}{2}r\pi\phi)}\sum_{r'} r'A_{r'}(-1)^{(r'-1)/2}$$
$$\times\{\Lambda_1(r,r') + (64\phi^4/\pi^2)\,\Lambda_3(r,r') + (64\phi^4/\pi^2)^2\,\Lambda_5(r,r') + \ldots\}, \tag{4.7.8a}$$

$$\beta_m = \frac{4(-1)^{(s-1)/2}\phi}{\pi(\lambda+\mu)\,\sigma(s\pi/2\phi)}\sum_{r'} r'A_{r'}(-1)^{(r'-1)/2}$$
$$\times\{\Lambda_0(r',s) + (64\phi^4/\pi^2)\,\Lambda_2(r',s) + (64\phi^4/\pi^2)^2\,\Lambda_4(r',s) + \ldots\}, \tag{4.7.8b}$$

with
$$\Lambda_0(r',s) = \left\{\frac{1}{s^2+\phi^2 r'^2} + \frac{2s^2 T(\frac{1}{2}r'\pi\phi)\,\chi(\frac{1}{2}r'\pi\phi)}{(s^2+\phi^2 r'^2)^2}\right\}, \tag{4.7.9a}$$

$$\Lambda_1(r,r') = \sum_s \frac{s\chi(s\pi/2\phi)}{(s^2+\phi^2 r^2)^2}\Lambda_0(r',s), \tag{4.7.9b}$$

$$\Lambda_2(r',s) = s^2\sum_r \frac{r^3\chi(\frac{1}{2}r'\pi\phi)}{(s^2+\phi^2 r^2)^2}\Lambda_1(r,r'), \tag{4.7.9c}$$

$$\Lambda_3(r,r') = \sum_s \frac{s\chi(s\pi/2\phi)}{(s^2+\phi^2 r^2)^2}\Lambda_2(r',s), \tag{4.7.9d}$$

$$\Lambda_4(r',s) = s^2\sum_r \frac{r^3\chi(\frac{1}{2}r'\pi\phi)}{(s^2+\phi^2 r^2)^2}\Lambda_3(r,r'). \tag{4.7.9e}$$

(iv) *Displacements*

$$u = \sum_n \left\{\frac{A_n E(nx)}{2\mu n E(\frac{1}{2}na)} + B_n\left(\left(\frac{\alpha}{2n} + \frac{a}{4\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)E(nx) - \frac{xe(nx)}{2\epsilon}\right)\right\}\sin ny$$
$$+\sum_m \beta_m\left\{\left(-\frac{1}{2m} + \frac{b}{4\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my) - \frac{yE(my)}{2\epsilon}\right\}\cos mx, \tag{4.7.10a}$$

$$v = \sum_n \left\{\frac{A_n e(nx)}{2\mu n E(\frac{1}{2}na)} + B_n\left(\left(-\frac{1}{2n} + \frac{a}{4\epsilon}\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx) - \frac{xE(nx)}{2\epsilon}\right)\right\}\cos ny$$
$$+\sum_m \beta_m\left\{\left(\frac{\alpha}{2m} + \frac{b}{4\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)E(my) - \frac{ye(my)}{2\epsilon}\right\}\sin mx. \tag{4.7.10b}$$

(v) *Stresses*

$$N_1 = (\lambda+\mu)\left\{\sum_n\left[\frac{A_n e(nx)}{(\lambda+\mu)E(\frac{1}{2}na)} + B_n\left(\left(1+\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx) - nxE(nx)\right)\right]\sin ny\right.$$
$$\left.+\sum_m \beta_m\left(\left(1-\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my) + myE(my)\right)\sin mx\right\}, \tag{4.7.11a}$$

$$N_2 = (\lambda+\mu)\left\{\sum_n\left[-\frac{A_n e(nx)}{(\lambda+\mu)E(\frac{1}{2}na)} + B_n\left(\left(1-\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}\right)e(nx) + nxE(nx)\right)\right]\sin ny\right.$$
$$\left.+\sum_m \beta_m\left(\left(1+\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right)e(my) - myE(my)\right)\sin mx\right\}, \tag{4.7.11b}$$

$$T_3 = (\lambda+\mu)\left\{\sum_n\left[\frac{A_n E(nx)}{(\lambda+\mu)E(\frac{1}{2}na)} + B_n\left(\tfrac{1}{2}na\frac{e(\frac{1}{2}na)}{E(\frac{1}{2}na)}E(nx) - nxe(nx)\right)\right]\cos ny\right.$$
$$\left.+\sum_m \beta_m\left(\tfrac{1}{2}mb\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}E(my) - mye(my)\right)\cos mx\right\}. \tag{4.7.11c}$$

(*h*) *Problem* SEB

Here

$$\left.\begin{aligned} &N_1 = N_2 = 0, \quad x = \pm\tfrac{1}{2}a \quad \text{and} \quad y = \pm\tfrac{1}{2}b, \\ &T_3 = f(y), \quad x = \tfrac{1}{2}a, \\ &T_3 = -f(y), \quad x = -\tfrac{1}{2}a, \\ &T_3 = 0, \quad y = \pm\tfrac{1}{2}b \end{aligned}\right\} \tag{4.8.1a–d}$$

with

$$f(y) = \sum_n A_n \cos ny. \tag{4.8.2}$$

(i) *Introduction of the series*

$$v_1 = \sum_n B_n E(nx) \sin ny, \quad n = \frac{r\pi}{b}; \quad r = 1, 3, 5, \ldots, \tag{4.8.3a}$$

$$v_2 = \sum_m \beta_m e(my) \cos mx, \quad m = \frac{2\pi p}{a}; \quad p = 1, 2, 3, \ldots, \tag{4.8.3b}$$

with

$$F_1 = -\frac{1}{2\epsilon} \sum_n \frac{1}{n} B_n xe(nx) \sin\, ny + \sum_n H_n E(nx) \sin ny, \tag{4.8.4a}$$

$$F_2 = -\frac{1}{2\epsilon} \sum_m \frac{1}{m} \beta_m yE(my) \cos mx + \sum_m G_m e(my) \cos mx. \tag{4.8.4b}$$

(ii) *Boundary conditions*

From $T_3 = f(y)$ at $x = \tfrac{1}{2}a$ we have

$$H_n = \frac{A_n}{2\mu n^2 e(\tfrac{1}{2}na)} + B_n \left[-\frac{1}{2n^2} + \frac{a}{4n\epsilon} \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right] \tag{4.8.5a}$$

and from $T_3 = 0$ at $y = \tfrac{1}{2}b$ we find

$$G_m = \beta_m \left[-\frac{1}{2m^2} + \frac{b}{4m\epsilon} \frac{e(\tfrac{1}{2}mb)}{E(\tfrac{1}{2}mb)} \right] \tag{4.8.5b}$$

which, as was the case with SEA, is exactly as for DOA, DOB.

From $N_1 = 0$ at $x = \tfrac{1}{2}a$

$$\sum_n \left\{ B_n \left(E(\tfrac{1}{2}na) + \tfrac{1}{2}na \frac{1}{e(\tfrac{1}{2}na)} \right) + \frac{A_n E(\tfrac{1}{2}na)}{(\lambda+\mu)\, e(\tfrac{1}{2}na)} \right\} \sin ny$$
$$+ \sum_m \beta_m \left\{ e(my) \left(1 - \tfrac{1}{2}mb \frac{e(\tfrac{1}{2}mb)}{E(\tfrac{1}{2}mb)} \right) + myE(my) \right\} \cos \tfrac{1}{2}ma = 0 \tag{4.8.6a}$$

and finally from $N_2 = 0$ on $y = \tfrac{1}{2}b$

$$\sum_n \left[B_n \left\{ E(nx) \left(1 - \tfrac{1}{2}na \frac{E(\tfrac{1}{2}na)}{e(\tfrac{1}{2}na)} \right) + nxe(nx) \right\} - \frac{A_n E(nx)}{(\lambda+\mu)\, e(\tfrac{1}{2}na)} \right] \sin \tfrac{1}{2}nb$$
$$+ \sum_m \left\{ \beta_m \left(e(\tfrac{1}{2}mb) - \tfrac{1}{2}mb \frac{1}{E(\tfrac{1}{2}mb)} \right) \right\} \cos mx = 0. \tag{4.8.6b}$$

(iii) *Coefficients* B_n, β_m

$$B_n = -\frac{A_n C(\frac{1}{2}na)}{(\lambda+\mu)\,\tau(\frac{1}{2}na)} - \frac{8n^2 \sin\frac{1}{2}nb}{b\tau(\frac{1}{2}na)} \sum_m \beta_m \frac{m}{(m^2+n^2)^2} E(\tfrac{1}{2}mb)\cos\tfrac{1}{2}ma, \quad (4.8.7\,a)$$

and

$$\beta_m = \frac{4\cos(\frac{1}{2}ma)}{(\lambda+\mu)\,\sigma(\frac{1}{2}mb)} \sum_n A_n \frac{n}{(m^2+n^2)} \sin\tfrac{1}{2}nb - \frac{8m^2\cos\frac{1}{2}ma}{a\sigma(\frac{1}{2}mb)} \sum_n B_n \frac{n}{(m^2+n^2)^2} e(\tfrac{1}{2}na)\sin\tfrac{1}{2}nb, \quad (4.8.7\,b)$$

from which we find

$$B_n = -\frac{A_n C(\frac{1}{2}r\pi\phi)}{(\lambda+\mu)\,\tau(\frac{1}{2}r\pi\phi)} - \frac{r^2(-1)^{(r-1)/2}\phi^4}{\pi^2(\lambda+\mu)\,\tau(\frac{1}{2}r\pi\phi)} \sum_{r'} r' A_{r'} (-1)^{(r'-1)/2} \times \{\Lambda_1(r,r') + (2\phi^4/\pi^2)\,\Lambda_3(r,r') + (2\phi^4/\pi^2)^2 \Lambda_5(r,r') + \ldots\}, \quad (4.8.8\,a)$$

and

$$\beta_m = \frac{(-1)^p \phi}{\pi(\lambda+\mu)\,\sigma(p\pi/\phi)} \sum_{r'} r' A_{r'} (-1)^{(r'-1)/2} \times \{\Lambda_0(p,r') + (2\phi^4/\pi^2)\,\Lambda_2(p,r') + (2\phi^4/\pi^2)^2 \Lambda_4(p,r') + \ldots\}, \quad (4.8.8\,b)$$

where

$$\Lambda_0(p,r') = \left\{\frac{1}{(p^2+\frac{1}{4}\phi^2 r'^2)} + \frac{2p^2 \Psi(\frac{1}{2}r'\pi\phi)\,C(\frac{1}{2}r'\pi\phi)}{(p^2+\frac{1}{4}\phi^2 r'^2)^2}\right\}, \quad (4.8.9\,a)$$

$$\Lambda_1(r,r') = \sum_p \frac{p\chi(p\pi/\phi)}{(p^2+\frac{1}{4}\phi^2 r^2)^2} \Lambda_0(p,r'), \quad (4.8.9\,b)$$

$$\Lambda_2(p,r') = p^2 \sum_r \frac{r^3 \Psi(\frac{1}{2}r\pi\phi)}{(p^2+\frac{1}{4}\phi^2 r^2)^2} \Lambda_1(r,r'), \quad (4.8.9\,c)$$

$$\Lambda_3(r,r') = \sum_p \frac{p\chi(p\pi/\phi)}{(p^2+\frac{1}{4}\phi^2 r^2)^2} \Lambda_2(p,r'), \quad (4.8.9\,d)$$

$$\Lambda_4(p,r') = p^2 \sum_r \frac{r^3 \Psi(\frac{1}{2}r\pi\phi)}{(p^2+\frac{1}{4}\phi^2 r^2)^2} \Lambda_3(r,r'). \quad (4.8.9\,e)$$

(iv) *Displacements*

$$u = \sum_n \left\{\frac{A_n e(nx)}{2\mu n e(\frac{1}{2}na)} + B_n\left(\left(\frac{\alpha}{2n} + \frac{a}{4\epsilon}\frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)}\right) e(nx) - \frac{xE(nx)}{2\epsilon}\right)\right\}\sin ny + \sum_m \beta_m \left\{\left(\frac{1}{2m} - \frac{b}{4\epsilon}\frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)}\right) e(my) + \frac{yE(my)}{2\epsilon}\right\}\sin mx, \quad (4.8.10\,a)$$

$$v = \sum_n \left\{ \frac{A_n E(nx)}{2\mu n e(\frac{1}{2}na)} + B_n \left(\left(-\frac{1}{2n} + \frac{a}{4\epsilon} \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} \right) E(nx) - \frac{x e(nx)}{2\epsilon} \right) \right\} \cos ny$$

$$+ \sum_m \beta_m \left\{ \left(\frac{\alpha}{2m} + \frac{b}{4\epsilon} \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) E(my) - \frac{y e(my)}{2\epsilon} \right\} \cos mx. \quad (4.8.10b)$$

(v) *Stresses*

$$N_1 = (\lambda+\mu) \left\{ \sum_n \left[\frac{A_n E(nx)}{(\lambda+\mu)\, e(\frac{1}{2}na)} + B_n \left(\left(1 + \tfrac{1}{2}na \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} \right) E(nx) - nxe(nx) \right) \right] \sin ny \right.$$

$$\left. + \sum_m \beta_m \left(\left(1 - \tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) e(my) + myE(my) \right) \cos mx \right\}, \quad (4.8.11a)$$

$$N_2 = (\lambda+\mu) \left\{ \sum_n \left[-\frac{A_n E(nx)}{(\lambda+\mu)\, e(\frac{1}{2}na)} + B_n \left(\left(1 - \tfrac{1}{2}na \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} \right) E(nx) + nxe(nx) \right) \right] \sin ny \right.$$

$$\left. + \sum_m \beta_m \left(\left(1 + \tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} \right) e(my) - myE(my) \right) \cos mx \right\}, \quad (4.8.11b)$$

$$T_3 = (\lambda+\mu) \left\{ \sum_m \left[\frac{A_n e(nx)}{(\lambda+\mu)\, e(\frac{1}{2}na)} + B_n \left(\tfrac{1}{2}na \frac{E(\frac{1}{2}na)}{e(\frac{1}{2}na)} e(nx) - nxE(nx) \right) \right] \cos ny \right.$$

$$\left. + \sum_m \beta_m \left(-\tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} E(my) + mye(my) \right) \sin mx \right\}. \quad (4.8.11c)$$

5. Stress boundary conditions and kinematic constraints

As a check on the preceding derivations, it is easy to show that all the solutions given in §§2 and 4 do indeed satisfy the stress boundary conditions as specified, as well as the obvious displacement constraints.

In the case of stresses, it is a simple matter to show explicitly that the shear stress conditions are automatically satisfied. Thus, for example, it is clear that T_3 in (4.3.11c) reduces to zero at $x = \pm\frac{1}{2}a$ since the coefficient of $\cos ny$ becomes zero and, of course, $\sin\frac{1}{2}ma = 0$; similarly, at $y = \pm\frac{1}{2}b$, $T_3 = 0$.

On the other hand, the direct stress conditions result in summations of series which, in general, cannot be reduced by inspection to the required boundary value. For instance, by reference to (4.3.11a), we find at $x = \frac{1}{2}a$ that

$$N_1 = (\lambda+\mu) \left\{ \sum_n B_n \tau(\tfrac{1}{2}na) \sin ny + \sum_m \beta_m (-1)^p \left[1 - \tfrac{1}{2}mb \frac{e(\frac{1}{2}mb)}{E(\frac{1}{2}mb)} e(my) + myE(my) \right] \right\}.$$

Now, one should note $(\lambda+\mu) B_n \tau(\frac{1}{2}na) = A_n$, which is the required result provided that the second term of (4.3.7a) (with the summation over n) as well as the complete second term of the above expression for N_1 (with the summation over m) reduces to zero. As one would expect, both these summations do reduce to zero for $x = \frac{1}{2}a$.

Table 2. *Kinematic constraints* (Z, *zero*; N, *non-zero*)

case	$u(0,0)$	$v(0,0)$	$\dfrac{\partial u}{\partial x}(0,0)$	$\dfrac{\partial u}{\partial y}(0,0)$	$\dfrac{\partial v}{\partial x}(0,0)$	$\dfrac{\partial v}{\partial y}(0,0)$
DEA	Z	Z	N	Z	Z	N
DEB	N	Z	Z	Z	Z	Z
DOA	Z	N	Z	Z	Z	Z
DOB	Z	Z	Z	N	N	Z
SOA	N	Z	Z	Z	Z	Z
SOB	Z	Z	N	Z	Z	N
SEA	Z	Z	Z	N	N	Z
SEB	Z	N	Z	Z	Z	Z

Clearly, in the general case of direct-stress boundary-conditions checks, the entire solution needs to be programmed, evaluating these conditions numerically. This we have done, all boundary values satisfying the appropriate conditions as required, although we emphasize that the computations were necessarily carried out at discrete values of ϕ. Of course, there are other obvious checks; for example, the explicit formulae always reduce to $N_1+N_2 = 2(\lambda+\mu)\,v$, as required.

When loading the plate, the appropriate conditions to examine are displacements at the centre ($u(0,0)$, $v(0,0)$) and the slopes and lines of symmetry of $x = 0$ and $y = 0$. For simplicity we tabulate the results given by the explicit formulae for $u, v, \partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, $\partial v/\partial y$ as either zero (Z) or non-zero (N) in table 2. When comparing with the diagrammatic stress applications for the eight cases, shown in figure 1, the correspondence is clearly correct.

6. Concluding remarks

The foregoing set of derivations provides exact closed-form formulae, albeit in the form of infinite series, for the stresses and displacements arising in a rectangular plate or plane-strain prism subjected to arbitrary loadings on its four edges. The desired result is obtained by the superposition of the relevant set of fundamental problems chosen from among the eight basic cases summarized herein. Thus, for the first time, the exact solution to two-dimensional elasticity problems with rectangular boundaries is available. This has been achieved by extending Mathieu's original pair of basic problems so as to encompass all eight possibilities.

Full convergence studies and applications to benchmark problems will be deferred to a separate article. However, it should be noted that computation is straightforward since, for most loadings, the series converge quickly, as can be seen from the form of the functions Λ_i throughout. Moreover, it was found in the previously mentioned applications (Pavlović & Baker 1988) that no more than four Λ functions were needed to compute the coefficients B_n and β_m.

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